Parity of ranks of abelian surfaces

Céline Maistret

University of Bristol

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Parity of ranks of abelian surfaces

1. Ranks and parity of ranks of abelian varieties
   - Conjectures
   - Example

2. Parity of ranks of principally polarized abelian surfaces

Ranks of abelian varieties and conjectures

Mordell-Weil Theorem

Let A/K be an abelian variety over a number field

$$A(K) \simeq \mathbb{Z}^{rk(A)} \oplus T, \quad rk_A, |T| < \infty.$$

Birch and Swinnerton-Dyer conjecture

Granting analytic continuation of the L-function of A/K to $\mathbb{C}$,

$$rk(A) = ord_{s=1} L(A/K, s) =: rk_{an}(A).$$

Conjectural functional equation

The completed L-function $L^*(A/K, s)$ satisfies

$$L^*(A/K, s) = W(A)\, L^*(A/K, 2-s), \quad W(A) \in \{\pm 1\}.$$

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Parity of analytic rank

Analytic rank

$$rk_{an}(A) := ord_{s=1} L(A/K, s).$$

Sign in functional equation

$$L^*(A/K, s) = W(A)\ L^*(A/K, 2-s), \quad W(A) \in \{\pm 1\}.$$

Consequence

$$(-1)^{rk_{an}(A)} = W(A).$$

Parity conjecture

B.S.D. modulo 2

$$(-1)^{rk(A)} \underset{BSD}{=} (-1)^{rk_{an}(A)} = W(A).$$

Global root number

The sign in the functional equation $W(A)$ is conjectured to be equal to the global root number of A:

$$W(A) = \omega(A).$$

Parity conjecture

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p^∞ Selmer rank and p-parity conjecture

p^∞ Selmer rank

For a prime p, define the p^∞ Selmer rank as

$$rk_p(A) = rk(A) + \delta_p, \text{ where}$$

$$\text{Ш}[p^\infty] = (\mathbb{Q}_p/\mathbb{Z}_p)^{\delta_p} \times \text{Ш}_0[p^\infty], \quad |\text{Ш}_0[p^\infty]| < \infty.$$

Assuming finiteness of $\text{Ш}(A)$; for all prime p

$$rk(A) = rk_p(A).$$

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Parity and isogeny

Theorem (Cassels). Isogeny invariance of B.S.D. quotient

Assuming $Ш(E)$ is finite, if $\Phi : E \to E'$ is an isogeny defined over $\mathbb{Q}$ then

$$\frac{|Ш(E)| Reg_E \, C_E}{|E(\mathbb{Q})_{tors}|^2} = \frac{|Ш(E')| Reg_{E'} \, C_{E'}}{|E'(\mathbb{Q})_{tors}|^2}$$

Example

$E/\mathbb{Q} : y^2 + y = x^3 + x^2 - 7x + 5,\quad E$ has a 3-isogeny,
$\Delta_E = -7 \cdot 13,\quad c_7 = c_{13} = 1,\quad c'_7 = c'_{13} = 3,\quad c_\infty = 3c'_\infty$

$$\Rightarrow \frac{Reg_E}{Reg_{E'}} = \frac{|Ш(E)||E'(\mathbb{Q})_{tors}|^2 C_E}{|Ш(E')||E(\mathbb{Q})_{tors}|^2 C_{E'}} = \frac{C_E}{C_{E'}} \cdot \square = \frac{3}{9} \cdot \square \neq 1$$

$\Rightarrow E$ has a point of infinite order.

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$\Rightarrow$ E has a point of infinite order.

Lemma (Dokchitser-Dokchitser)

If Φ is an isogeny of degree d such that $\Phi^*\Phi = \Phi\Phi^* = [d]$ then

$$\frac{Reg_E}{Reg_{E'}} = d^{rk(E)} \cdot \square$$

$c_7 = c_{13} = 1, \quad c'_7 = c'_{13} = 3, \quad c_\infty = 3c'_\infty$

$$\frac{Reg_E}{Reg_{E'}} = \frac{1}{3} \cdot \square = 3^{rk(E)} \cdot \square$$

$\Rightarrow E$ has odd rank

Remark

Without assuming finiteness of $Ш(E)$, can prove $rk_3(E)$ is odd.

"Proof" of parity conjecture for p.p. abelian surfaces

- Enough to prove 2-parity conjecture for Jacobians of genus 2 curves

$$C : y^2 = f(x), \qquad deg(f) = 6, \qquad Gal(f) \subseteq C_2 \times D_4.$$

- Computation of parities

$$(-1)^{rk_2(J)} = \prod_v (-1)^{ord_2(\frac{c_v m_v}{c'_v m'_v})}, \qquad \omega(J) = \prod_v \omega_v(J).$$

- 2-parity conjecture

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- Enough to consider Jacobians of genus 2 curves

$$C : y^2 = f(x), \qquad deg(f) = 6$$

Theorem (cf Gonzales-Guàrdia-Rotger)

Let A/K be a principally polarized abelian surface defined over a number field K. Then A is one of the following three types:

- $A \simeq_K J(C)$, where C/K is a smooth curve of genus 2,
- $A \simeq_K E_1 \times E_2$, where E_1, E_2 are two elliptic curves defined over K,
- $A \simeq_K Res_{F/K} E$, where $Res_{F/K} E$ is the Weil restriction of an elliptic curve defined over a quadratic extension F/K.

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$$C : y^2 = f(x), \qquad deg(f) = 6, \qquad Gal(f) \subseteq C_2 \times D_4.$$

2-parity conjecture

If $Gal(f) \subseteq C_2 \times D_4$ then J admits a **Richelot isogeny** Φ s.t. $\Phi\Phi^* = [2]$.

2-parity conjecture to parity conjecture

$$Ш[2^\infty] \text{ finite, then 2-parity conjecture} \Rightarrow \text{ parity conjecture.}$$

Removing conditions on $Gal(f)$: Regulator constants

Suppose $C : y^2 = f(x)$ is semistable,
• $K_f =$ splitting field of f,
• $Ш(J/K_f)[p^\infty]$ is finite for $p = 3, 5$,
• Parity conjecture holds for J/L for all $K \subseteq L \subseteq K_f$ with $Gal(K_f/L) \subseteq C_2 \times D_4$.
Then the parity conjecture holds for J/K.

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"Proof" of parity conjecture for p.p. abelian surfaces

- Enough to prove 2-parity conjecture for Jacobians of genus 2 curves

$$C : y^2 = f(x), \qquad deg(f) = 6, \qquad Gal(f) \subseteq C_2 \times D_4.$$

- Computation of parities

$$(-1)^{rk_2(J)} = \prod_v (-1)^{ord_2(\frac{c_v m_v}{c'_v m'_v})}, \qquad \omega(J) = \prod_v \omega_v(J).$$

- 2-parity conjecture

$$(-1)^{rk_2(J)} = \prod_v (-1)^{ord_2(\frac{c_v m_v}{c'_v m'_v})} = \prod_v \omega_v(J) = \omega(J)$$

- Computation of parities

Lemma

Let $\Phi : J \to J'$ be an isogeny satisfying $\Phi^*\Phi = \Phi\Phi^* = [2]$. Then

$$2^{rk_2(J)} = \frac{C_J}{C_{J'}} \frac{|\text{Ш}_0(J)[2^\infty]|}{|\text{Ш}_0(J')[2^\infty]|} \square$$

Theorem 2.i

Assume that $Gal(f) \subseteq C_2 \times D_4$. Then

$$(-1)^{rk_2(J)} = \prod_v (-1)^{ord_2(\frac{c_v m_v}{c'_v m'_v})},$$

where c_v, c'_v denote the Tamagawa numbers of J and J' respectively and $m_v = 2$ if C is deficient at v, $m_v = 1$ otherwise (cf Poonen-Stoll).

- Computation of parities

Theorem 2.i

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where c_v, c'_v denote the Tamagawa numbers of J and J' respectively and $m_v = 2$ if C is deficient at v, $m_v = 1$ otherwise (cf Poonen-Stoll).

Joint with T. and V. Dokchitser, A. Morgan, and Alexander Betts :

Local arithmetic of hyperelliptic curves

Let $K/\mathbb{Q}_p$ finite, p odd. Then for J/K semistable, c_v, m_v and ω_v are computable.

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- 2-parity conjecture

Theorem

If $Gal(f) \subseteq C_2 \times D_4$ and C is semistable at v (and "lovely" if $v \mid 2$, cf. Adam Morgan) then

$$(-1)^{ord_2\left(\frac{c_v m_v}{c'_v m'_v}\right)} = H_v \cdot \omega_v,$$

where H_v is a product of Hilbert symbols at v.

- 2-parity conjecture

Corollary (Theorem 2.ii.)

Assume that $Gal(f) \subseteq C_2 \times D_4$, that C is semistable and "lovely" if $v \mid 2$. Then

$$\prod_v (-1)^{ord_2\left(\frac{c_v m_v}{c'_v m'_v}\right)} = \prod_v \omega_v(J).$$

By Theorem 2.i. the 2-parity conjecture holds as $(-1)^{rk_2(J)} = \omega(J)$.

Parity of $rk(J)$

Theorem 1 (joint with V. Dokchitser)

Assume that C/K is semistable and "lovely" at 2-adic places. If $Ш(J/K_f)[p^\infty]$ is finite for $p = 2, 3, 5$ then the parity conjecture holds for J/K, i.e.

$$(-1)^{rk(J)} = \omega(J).$$

Thank you for your attention

"Lovely" at 2-adic places v

$$\mathcal{F} : y^2 = (x^2 - (4t_1)^2)(x^2 + t_2x + t_3)(x^2 + t_4x + t_5),$$

where $t_1 \in \mathcal{O}_{K_v}$, $\quad t_2 \equiv 1 \mod 2$, $\quad t_3 = \frac{1}{4} + 2z, z \in \mathcal{O}_{K_v}$,
$t_4 = -2 \mod 8$, $\quad t_5 \equiv 1 \mod 8$.

Theorem

Fix an exterior form ω' of J' and denote $\omega_v'^o$, ω_v^o the Néron exterior forms at the place v of K associated to ω' and $\phi^*\omega'$ respectively. Then $(-1)^{rk_2(J)} = \prod_v (-1)^{\lambda_v}$ with

$$\lambda_{v|\infty} = ord_2\left(\frac{n \cdot m_v}{|ker(\alpha)| \cdot n' \cdot m_v'}\right), \quad \lambda_{v\nmid\infty} = ord_2\left(\frac{c_v \cdot m_v}{c_v' \cdot m_v'}\left|\frac{\phi^*\omega_v'^o}{\omega_v^o}\right|_v\right),$$

where n, n' are the number of K_v-connected components of J and J', α is the restriction of ϕ to the identity component of $J(K_v)$, c_v and c_v' the Tamagawa numbers of J and J', and $m_v = 2$ if C is deficient at v, $m_v = 1$ otherwise.

Theorem: Regulator constants (T. and V. Dokchitser)

Suppose

- A semistable p.p. abelian variety,
- $F = K(A[2])$,
- $Ш(A/F)[p^\infty]$ is finite for odd primes p dividing $[F : K]$,
- Parity holds for A/L for all $K \subseteq L \subseteq F$ with $Gal(F/L)$ a 2-group.

Then the parity conjecture holds for A/K.

Remark

The Sylow 2-subgroup of S_6 is $C_2 \times D_4$.
Hence if $Gal(K_f/L)$ is a 2-group then $Gal(K_f/L) \subseteq C_2 \times D_4$.
By Theorem 2.ii: if $Gal(K_f/L) \subseteq C_2 \times D_4$, C semistable and "lovely" at 2-adic places then the 2-parity conjecture holds for J/L.
Thus if $|Ш(J/K_f)[2^\infty]| < \infty$ then the parity conjecture holds for J/L.

- $\mathrm{Gal}(f) \subseteq C_2 \times D_4 \implies$ Richelot isogeny

$$f(x) = q_1(x)q_2(x)q_3(x) \text{ with roots } \alpha_i, \beta_i.$$

$$D_1 = [(\alpha_1, 0), (\beta_1, 0)], \quad D_2 = [(\alpha_2, 0), (\beta_2, 0)], \quad D_3 = [(\alpha_3, 0), (\beta_3, 0)]$$

lie in $J(\overline{K})[2]$ and $\{0, D_1, D_2, D_3\}$ is a Galois stable subgroup of $J(K)[2]$.

Proposition

If $Gal(f) \subseteq C_2 \times D_4$ then J admits a **Richelot isogeny** Φ s.t. $\Phi\Phi^* = [2]$.

Remark : Explicit construction

There is an explicit model for the curve C' underlying the isogenous Jacobian J'.

Local comparison

Let $f(x) = q_1(x)q_2(x)q_3(x)$ with roots $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3$ and $\Delta_i = Disc(q_i(x))$.

$$H_v = (-1, I_{22}I_{41}I_{43}I_{60})_v(I_{20}, -I_{40}I_{44})_v(I_{40}, \ell I_{60}I_{43})_v(c, I_{23}I_{44}I_{80})_v(I_{23}, I_{41})_v$$

$$(I_{45}, -\ell I_{22}I_{21})_v(I_{44}, 2I_{22}I_{42}I_{43})_v(I_{80}, -2I_{41}I_{42}I_{60})_v(I_{42}, -I_{60}I_{43})_v,$$

where....

$I_{20} = \frac{1}{2^3}(\Delta_2 + \Delta_3)$,
$I_{21} = (\alpha_2 + \beta_2)(\alpha_3 + \beta_3)$,
$I_{22} = (\alpha_2 - \alpha_3)(\beta_2 - \beta_3) + (\beta_2 - \alpha_3)(\alpha_2 - \beta_3)$,
$I_{23} = \Delta_1$,
$I_{40} = \frac{1}{2^6}(\Delta_2 - \Delta_3$,
$I_{41} = 8((\alpha_2 - \beta_1)(\beta_2 - \beta_1)(\alpha_3 - \beta_1)(\beta_3 - \beta_1) + (\alpha_2 - \alpha_1)(\beta_2 - \alpha_1)(\alpha_3 - \alpha_1)(\beta_3 - \alpha_1))$,
$I_{42} = ((\alpha_2 - \alpha_1)(\alpha_2 - \beta_1) + (\beta_2 - \alpha_1)(\beta_2 - \beta_1))((\alpha_3 - \alpha_1)(\alpha_3 - \beta_1) + (\beta_3 - \alpha_1)(\beta_3 - \beta_1))$,
$I_{43} = \Delta_2((\alpha_2 - \alpha_1)(\alpha_2 - \beta_1) + (\beta_2 - \alpha_1)(\beta_2 - \beta_1)) + \Delta_3((\alpha_3 - \alpha_1)(\alpha_3 - \beta_1) + (\beta_3 - \alpha_1)(\beta_3 - \beta_1))$,
$I_{44} = \Delta_2\Delta_3$,
$I_{45} = 4(\beta_3 - \beta_2)(\alpha_3 - \beta_2)(\alpha_2 - \beta_3)(\alpha_2 - \alpha_3)$,
$I_{60} = (\alpha_2 - \alpha_1)(\alpha_2 - \beta_1)(\beta_2 - \alpha_1)(\beta_2 - \beta_1)((\alpha_2 - \alpha_1)(\alpha_2 - \beta_1) + (\beta_2 - \alpha_1)(\beta_2 - \beta_1)) + (\alpha_3 - \alpha_1)(\alpha_3 - \beta_1)(\beta_3 - \alpha_1)(\beta_3 - \beta_1)((\alpha_3 - \alpha_1)(\alpha_3 - \beta_1) + (\beta_3 - \alpha_1)(\beta_3 - \beta_1))$,
$I_{80} = (\alpha_2 - \alpha_1)(\alpha_2 - \beta_1)(\beta_2 - \alpha_1)(\beta_2 - \beta_1)(\alpha_3 - \alpha_1)(\alpha_3 - \beta_1)(\beta_3 - \alpha_1)(\beta_3 - \beta_1)$.

Known results for the parity conjecture

From work of Monsky, Nekovar, Dokchitser and Dokchitser, Cesnavicius, Coates-Fukaya-Kato-Sujatha, Kramer-Tunnell, Morgan

- $E/\mathbb{Q}$ assuming $Ш(E)[p^\infty]$ finite for some p,
- E/K, for a totally real field K, assuming $Ш(E)[p^\infty]$ finite for some p (+ mild constraints),
- E/K admitting a p-isogeny, assuming $Ш(E)[p^\infty]$ finite,
- $E/K(\sqrt{d})$, E defined over K, $d \in K^\times \setminus K^{\times 2}$, $Ш(E/K(\sqrt{d}))[2^\infty]$ finite,
- E/K, $Ш(E/F)[2^\infty], Ш(E/F)[3^\infty]$, $F = K(E[2])$.
- $A/K(\sqrt{d})$, $A = Jac(C)$, C semistable hyperelliptic curve over K, $d \in K^\times \setminus K^{\times 2}$, $Ш(A/K(\sqrt{d}))[2^\infty]$ finite, (+ mild constraints),
- p.p. Abelian varieties A/K admitting an isogeny $\Phi : A \to A'$ s.t. $\Phi^*\Phi = [p]$, $Ш(A)[p^\infty]$ finite, p *odd* (+ mild constraints),